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Differential cohomology of C^{∞} ($\mathfrak{G}^*$)

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Abstract. *Let $\mathfrak{G}$ be a finite dimensional real Lie algebra and $\mathfrak{G}^*$ its dual. $\mathfrak{G}^*$ is a Poisson manifold. Thus the space $C^{\infty}(\mathfrak{G}^*)$ of C^{∞} functions on $\mathfrak{G}^*$ has an associative and a Lie algebra structure. The problem of formal deformations of such a structure needs the determination of some cohomology groups of $C^{\infty}(\mathfrak{G}^*)$, considered as a module on itself for left multiplication or adjoint representation. We determine here these groups. The result is very similar to the case of $C^{\infty}(W)$, where W is a symplectic manifold except for the Lie algebras $h_r \times \mathbb{R}^m$, direct products of Heisenberg and abelian Lie algebras.*

INTRODUCTION

Deformations of the associative – and – Lie algebra of the C^{∞} functions on a symplectic or Poisson manifold were introduced by M. Flato and A. Lichnerowicz [1] in order to define an autonomous quantization theory without need for Hilbert space operators. These deformations were called (differential) * products. We consider here only differential * products, thus we shall omit the term «differential» from now on.

The problem of existence and equivalence of * products on a symplectic manifold is now entirely solved: first the corresponding differential cohomologies were determined by J. Vey [2] (for the Hochschild cohomology, corresponding

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to associative deformations) and S. Gutt [3] (for the three first groups in the Chevalley cohomology, corresponding to deformations of Lie algebras structures). Using explicitely these groups, M. Cahen and S. Gutt [4] (in a special case) and M. de Wilde and P. Lecomte [5] proved the existence of * products for any symplectic manifold W. The classification of these * products was considered by O.M. Neroslavsky and A.T. Vlassov [6] and A. Lichnerowicz [7]: the second de Rham cohomological group of W determines that classification.

Let us consider now a Poisson manifold W. In [8], A. Lichnerowicz solved the problem of existence and classification of «tangential» * products on W, if W is regular i.e. if all its symplectic leaves have the same dimension and if the tangential corresponding cohomological groups vanish. This result was generalized by F. Guidera in [9].

Among the non regular Poisson manifold, there are some very particular and interesting: the dual $\mathfrak{G}^*$ of a finite dimensional real Lie algebra. The corresponding symplectic leaves are the orbits of the coadjoint action ([10], [8]) thus $\mathfrak{G}^*$ is never a regular Poisson manifold, except if $\mathfrak{G}$ is abelian.

For $\mathfrak{G}^*$, the problem of existence of * products was solved by S. Gutt [11]. In fact in [11], S. Gutt build a * product on the symplectic manifold T^*G of any Lie group, the «vertical» part of this * product is a * product on $\mathfrak{G}^*$. Unfortunately this * product is in general not tangential ([12], [13]) and a nilpotent example (called $\mathfrak{G}_{5,4}$ by Dixmier) was found where no tangential * products exist on * [14], moreover the explicit tangential products considered in [15] (and [14]) in the semi simple case are not differential * products in general.

In order to generalize the methods of [4] and [5] to the problem of existence and classification of * products on $\mathfrak{G}^*$, we have to consider the (differential) Hochschild and Chevalley cohomologies on $\mathfrak{G}^*$. The first one does not depend of the structure of $\mathfrak{G}$, it was completely determined by J. Vey [2] and R. Berger [16], the result is exactly the same as Vey's result for symplectic manifold. The second one was never considered until now. Thus we determine here the first and second groups in this cohomology, following the method of S. Gutt in [3] (for symplectic manifold). Of course, the proofs are entirely new since the structure tensor is here not inversible: in [3] that tensor inverse is used at each step. But the result is generally very similar except if $\mathfrak{G}$ is the direct product $h_r \times \mathbb{R}^m$ of a Heisenberg and an abelian Lie algebra (one of these factors can be trivial).

1. DEFINITIONS AND NOTATIONS

$\mathfrak{G}$ is a finite dimensional real Lie algebra, $\mathfrak{G}^*$ its dual.

$S(\mathfrak{G})$ is the symmetric algebra of $\mathfrak{G}$ with gradation $(S^m), m \geqslant 0$, $S_+ = \sum_{m>0} S^m$,

$S^0_{++} = \sum_{m>1} S^m$.

$X_1, \ldots, X_n$ is a basis of $\mathfrak{G}$. We shall choose it in §4. The duality: $\langle X, \xi \rangle$ between $\mathfrak{G}$ and $\mathfrak{G}^*$ allows us to consider X_i as a function, denoted x_i on $\mathfrak{G}^*$. Then $S(\mathfrak{G})$ is the space of polynomial functions in the variables x_i and $C^\infty(\mathfrak{G}^*)$ the space of C^∞ functions in the variables x_i. We shall denote by N one of these spaces.

The structure constants C^k_{ij} of $\mathfrak{G}$ are defined by:

$$[X_i, X_j] = \sum_k C^k_{ij} X_k .$$

We denote by ∂_i the derivation $\dfrac{\partial}{\partial x_i}$ of N and if α is a multi index by D^α the differential operator:

$$D^\alpha = (\partial_1)^{a_1} \ldots (\partial_n)^{a_n} \text{ if } \alpha = (a_1, \ldots, a_n) \quad a_i \in \mathbb{N}.$$

We denote by $[i]$ the multi index $(0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is at the i^{th} place, $|\alpha|$, $\alpha!$ have the usual meaning.

The Poisson structure of * is given by the 2-tensor Λ such that:

$$i(\Lambda)(X \wedge Y)_\xi = \langle [X, Y], \xi \rangle$$

or by the Poisson bracket:

$$\{u, v\} = \sum_{ijk} C^k_{ij} x_k \partial_i u \, \partial_j v \qquad \forall u, v \in N$$

(we shall denote $\{x_i, x_j\}$ by $[x_i, x_j]$). $(N, \{\ \})$ is now an infinite dimensional Lie algebra. A multi differential operator C on N is a p-linear application from $N \times \ldots \times N$ to N such that:

$$C(u_1, \ldots, u_p) = \sum_{\alpha_1 \ldots \alpha_p} C_{\alpha_1 \ldots \alpha_p} D^{\alpha_1} u_1 D^{\alpha_2} u_2 \ldots D^{\alpha_p} u_p$$

where $C_{\alpha_1 \ldots \alpha_p}$ are all in N and the sum is finite. We shall denote it by

$$C = \sum_{\alpha_1 \ldots \alpha_p} C_{\alpha_1 \ldots \alpha_p} D^{\alpha_1} \cup D^{\alpha_2} \cup \ldots \cup D^{\alpha_p}$$

for instance:

$$\{,\} = \sum_{i,j} [x_i, x_j] D^{[i]} \cup D^{[j]}.$$

Finally we shall denote by I the space N^G of invariant polynomials (resp. C^∞ functions) on $\mathfrak{G}^*$:

$$\lambda \in I \quad \text{if and only if} \quad \{x, \lambda\} = 0 \qquad \forall X \in \mathfrak{G}.$$

I is the isotypic component of the trivial representation in the representation π of $\mathfrak{G}$ on N defined by:

$$\pi(X)u = \{x, u\} \qquad\qquad \forall u \in N,\ X \in \mathfrak{G}.$$

2. HOCHSCHILD COHOMOLOGY OF N

We consider here only differential, vanishing on constants cochains. i.e. a p cochain is a p differential operator C:

$$C = \sum_{\alpha_1 \ldots \alpha_p \text{ (finite)}} C_{\alpha_1 \ldots \alpha_p} D^{\alpha_1} \cup \ldots \cup D^{\alpha_p}$$

such that $C(u_0, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_p) = 0$ $\forall i$, or $|\alpha_i| > 0$ $\forall i$. We shall denote by $C^p(N)$ the space of p-cochain and the coboundary operator δ is given by:

$$(\delta C)(u_0, ..., u_p) = u_0 C(u_1, ..., u_p) - C(u_0 \cdot u_1, u_2, ..., u_p) + ...$$
$$+ (-1)^i C(u_0, ..., u_{i-1}, u_i \cdot u_{i+1}, ..., u_p) + ... + (-1)^{p-1} C(u_0, ..., u_{p-1}) \cdot u_p.$$

The space of p cocycle $Z^p_{\text{diff}}(N, \delta)$ is the kernel of δ in $C^p(N)$, the space of p coboundary $B^p_{\text{diff}}(N, \delta)$ is $\delta(C^{p-1}(N))$ and the cohomology group $H^p_{\text{diff}}(N, \delta)$ is the quotient $Z^p_{\text{diff}}(N, \delta)/B^p_{\text{diff}}(N, \delta)$. These groups were determined by J. Vey [2] for $N = C^\infty(\mathfrak{G}^*)$ and R. Berger [16] for $N = S(\mathfrak{G})$. They found:

THEOREM 1.

$$H^p_{\text{diff}}(N, \delta) \simeq Hom(\Lambda^p \mathfrak{G}, N).$$

The isomorphism being given by:
To $[C] \in H^p_{\text{diff}}(N, \delta)$, *we associate:* $c \in Hom(\Lambda^p \mathfrak{G}, N)$ *defined by*

$$c(X_1, \ldots, X_p) = \sum_{\sigma \in \mathfrak{G}_p} \epsilon(\sigma) C(x_{\sigma(1)}, \ldots, x_{\sigma(p)})$$

$\mathfrak{G}_p$ *is the group of permutation of* $\{1, \ldots, p\}$). ∎

3. CHEVALLEY COHOMOLOGY OF N

This cohomology is associated to the question of deformations of $(N, \{\ \})$.

It was introduced by Chevalley and Eilenberg. Here N is the Lie algebra view as a module on itself for the adjoint representation.

The p-cochains are completely antisymmetric elements of $C^p(N)$.

We denote their space by $C^p_a(N)$, the coboundary operator ∂ is defined by:

$$(\partial C)(u_0, ..., u_p) = \sum_{i=0}^{p} (-1)^i \{u_i, C(u_0, ..., \hat{u}_i, ..., u_p)\}$$

$$- \sum_{0 \leqslant i<j \leqslant p} (-1)^{i+j} C(\{u_i, u_j\}, u_0, ..., \hat{u}_i, ..., \hat{u}_j, ..., u_p)$$

where $\hat{u}_i$ means that this argument is missing.

$Z^p_{\text{diff}}(N, \partial)$ is its kernel, $B^p_{\text{diff}}(N, \partial)$ the space $\partial(C^{p-1}_a(N))$, $H^p_{\text{diff}}(N, \partial)$ the quotient $Z^p_{\text{diff}}(N, \partial)/B^p_{\text{diff}}(N, \partial)$.

If $\mathfrak{G}$ is abelian, ∂ vanishes thus we have the following.

PROPOSITION 1. *If $\mathfrak{G}$ is the abelian n-dimensional Lie algebra:*

$$H^p_{\text{diff}}(N, \partial) \simeq (\Lambda^p S_+) \otimes N$$

where S_+ is the ideal generated by $\mathfrak{G}$ in $S(\mathfrak{G})$ and the isomorphism is given by:

$$[C] \to \sum_{\alpha_1 \ldots \alpha_p} (x^{\alpha_1} \wedge \ldots \wedge x^{\alpha_p}) \otimes C_{\alpha_1 \ldots \alpha_p}$$

where $x^\alpha = x_1^{a_1} \ldots x_n^{a_n}$ if $\alpha = (a_1 \ldots a_n)$ and if

$$C = \sum_{\alpha_1 \ldots \alpha_p} C_{\alpha_1 \ldots \alpha_p} D^{\alpha_1} \cup \ldots \cup D^{\alpha_p}.$$

■

See for instance [17] for definitions of basis in $\Lambda^p S_+$.

From now on, we shall suppose $\mathfrak{G}$ non abelian.

In order to compute $H^1_{\text{diff}}(N, \partial)$ and $H^2_{\text{diff}}(N, \partial)$, we have to choose the basis of $\mathfrak{G}$ and an order on p-uple of multi indexes.

4. TECHNICAL PRELIMINARIES

LEMMA 1. *1) If* $\dim[\mathfrak{G}, \mathfrak{G}] > 1$, *we choose a basis* $(X_1 \ldots X_n)$ *in* $\mathfrak{G}$ *such that:*

a) $\quad [X_1, X_i] \neq 0 \qquad \forall i = 2, \ldots, n;$

b) $$\mathbb{R}[X_1, X_2] \cap \left(\sum_{i=3}^{n} \mathbb{R}[X_1, X_i] \right) = \{0\}.$$

2) If $\dim[\mathfrak{G}, \mathfrak{G}] = 1$, *we choose* $(X_1 \ldots X_n)$ *such that* $\mathbb{R}[X_1, X_2] = [\mathfrak{G}, \mathfrak{G}]$. *Two cases happen:*

a) If $[X_1, X_2]$ *is central,* $\mathfrak{G}$ *is the direct product* $h_r \times \mathbb{R}^m$ of a $2r+1$ *dimensional Heiseberg Lie algebra* $(r > 0)$ *by a* m *dimensional abelian Lie algebra,* $(m \geqslant 0)$.

b) If $[X_1, X_2]$ *is not central, we choose* X_1, X_2 *such that* $[X_1, X_2] = X_2$ *and* $\sum_{i=2}^{n} \mathbb{R} X_i$ *is an algebra* $h_r \times \mathbb{R}^m$ *(here r can be* 0 *thus* $h_0 = \{0\}$*), commuting with* X_2.

Proof. In the first case there exists two vectors $[X, Y]$, $[Z, T]$ linearly independent in $[\mathfrak{G}, \mathfrak{G}]$. Let us prove that we can choose $Z = X$. If $[Z, Y]$ and $[Z, T]$ (resp. $[X, Y]$ and $[Z, Y]$, resp. $[X, Y]$ and $[X, T]$, resp. $[X, T]$ and $[Z, T]$) are independent, it is true. If it is not the case $[Z, Y]$ and $[X, T]$ vanish, thus $[X + Z, Y] = [X, Y]$ and $[X + Z, T] = [Z, T]$ are independent. We suppose now $[X_1, X_2]$ and $[X_1, X_3]$ linearly independent, thus (X_1, X_2, X_3) are independent, we choose a supplementary V of $\mathbb{R}[X_1, X_2]$ in $[\mathfrak{G}, \mathfrak{G}]$ such that $[X_1, X_3] \in V$ and a basis $(X_1, X_2, X_3, e_4 \ldots e_n)$ of $\mathfrak{G}$. We have:

$$[X_1, e_i] = a_i[X_1, X_2] + v_i \quad \text{where } a_i \in \mathbb{R},\ v_i \in V,$$

we put:

$$X_i = e_i - a_i X_2 + b_i X_3 \quad \text{where } b_i \in \mathbb{R} \text{ is such that}$$

$$[X_1, X_i] = v_i + b_i[X_1, X_3] \neq 0.$$

In the second case, if $[\mathfrak{G}, \mathfrak{G}]$ is generated by the central element

$$X = [X_1, X_2],$$

the space spanned by (X, X_1, X_2) is three dimensional and if $(X, X_1, X_2, e_3, \ldots, e_{n-1})$ is a basis of $\mathfrak{G}$, the basis: $(X, X_1, X_2, e'_3, \ldots, e'_{n-1})$ where:

$$e'_i = e_i - a_i X_2 + b_i X_1 \quad \text{if } [X_1, e_i] = a_i X,\ [X_2, e_i] = b_i X$$

satisfies:

$$\mathfrak{G}' = \mathbb{R}\, X \oplus \Sigma\, \mathbb{R}\, e'_i$$

commutes with X_1 and X_2, $[\mathfrak{G}', \mathfrak{G}']$ is either nul or one dimensional generated by X central, the conclusion follows by induction.

If $[\mathfrak{G}, \mathfrak{G}]$ is generated by X non central, we can choose X_1, X_2 such that:

$$[X_1, X_2] = X_2 = X$$

and with the same argument find a basis $(X_1, X_2, e'_3, \ldots, e'_n)$ satisfying: $[X_1, e'_i] = [X_2, e'_i] = 0$ and if $\mathfrak{G}' = \mathbb{R}\, X \oplus \sum_{i=3}^{n} \mathbb{R}\; e'_i$, $[\mathfrak{G}', \mathfrak{G}']$ is either $\{0\}$ or $\mathbb{R}\, X$ this proves our lemma. ∎

DEFINITION. Let $A = (\alpha_1, \ldots, \alpha_p)$ and $B = (\beta_1, \ldots, \beta_p)$ two p-uple of multi indices.

We shall say that

$$(\alpha_1, \ldots \alpha_p) < (\beta_1, \ldots, \beta_p),$$

if and only if:

either $$\sum_{i=1}^{p} |\alpha_i| < \sum_{i=1}^{p} |\beta_i|$$

or if $\sum_i |\alpha_i| = \sum_i |\beta_i|$, if $|\alpha_1| = |\beta_1|, \ldots, |\alpha_{j-1}| = |\beta_{j-1}|, |\alpha_j| < |\beta_j|$ or if $|\alpha_i| = |\beta_i|\ \forall i$, if $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_{j-1} = \beta_{j-1}, \alpha_j < \beta_j$ where $\alpha_j < \beta_j$ means if $\alpha_j = (a_1, \ldots, a_n), \beta_j = (b_1, \ldots, b_n)$:

$$a_1 = b_1, \ldots, a_{\ell-1} = b_{\ell-1}, a_\ell < b_\ell.$$

We denote by $A \leqslant B$ the relation $A < B$ or $A = B$. We call order of A the multi index: $\sigma(A) = (|\alpha_1|, \ldots, |\alpha_p|)$. ∎

$<$ is now a total order on the set of p-uple of multi indices.

Let us remark that for $p = 1$ we define an order on multi indices and if i and j are in $\{1, \ldots, n\}$, $i < j$ means $[j] < [i]$.

DEFINITION. Let $\alpha = (a_1, \ldots, a_n)$ a multi index, $r \in \mathbb{N}$, we denote by: $\gamma_r(\alpha) = \inf \gamma$ such that $|\gamma| = r$ and $\alpha - \gamma$ «is a multi index» i.e. we consider the smallest multi index $\gamma = (c_1, \ldots, c_n)$ such that $c_1 + \ldots + c_n = r$ and $a_i - c_i \geqslant 0$ $\forall i$. We shall denote this last condition by $(\alpha - \gamma) > 0$.

We can add, substract, define support of multi indices ... when we consider these as functions from $\{1, \ldots, n\}$ to $\mathbb{Z}$. ∎

LEMMA 2. *Let s, r be two integers such that $s \geqslant r > 0$, E_s a set of multi indices α such that $|\alpha| = s$. We put: $M(E_s, \mathfrak{G}, r) = \{(\delta, \epsilon)$ where δ, ϵ are multi indices such that:*

$$\delta = (\alpha - \gamma) + [i], \quad \epsilon = \gamma + [j] \quad \textit{where}$$

$\alpha \in E_s$, γ is a multi index, $|\gamma| = r$, $(\alpha - \gamma) > 0$ and $[X_i, X_j] \neq 0\}$. Then the biggest element of $M(E_s, \mathfrak{G}, r)$ is:

$$(\delta_0, \epsilon_0) = ((\alpha_0 - \gamma_r(\alpha_0)) + [1],\ \gamma_r(\alpha_0) + [2]) \quad \text{where } \alpha_0 = \operatorname{Sup} E_s$$

and if (δ_0, ϵ_0) can be written:

$$(\delta_0, \epsilon_0) = ((\alpha - \gamma) + [i],\ \ \gamma + [j])$$

where $\alpha \in E_s$, γ is a multi index, $|\gamma| = r$, $(\alpha - \gamma) > 0$ and $[X_i, X_j] \neq 0$, then:

$$\alpha = \alpha_0, \quad \gamma = \gamma_r(\alpha_0), \quad i = 1, \quad j = 2.$$

Proof. Let us suppose $r = 1$, of course:

$$\operatorname{Sup}\{(\alpha - [j]) + [i] \quad i, j \in \{1, \ldots, n\}\} = (\alpha - \gamma_1(\alpha)) + [1]$$

and $(\alpha - [j]) + [i] = (\alpha - \gamma_1(\alpha)) + [1]$ implies $[j] = \gamma_1(\alpha)$, $i = 1$.

Now if $\alpha \leqslant \alpha'$, we shall prove that:

$$((\alpha - \gamma_1(\alpha)) + [1],\ \gamma_1(\alpha) + [2]) \leqslant ((\alpha' - \gamma_1(\alpha')) + [1],\ \gamma_1(\alpha') + [2])$$

the equality holding only if $\alpha = \alpha'$. In fact, if: $\alpha = (a_1, \ldots, a_\ell, 0, \ldots, 0)$ $\alpha' = (a'_1, \ldots, a'_{\ell'}, 0, \ldots, 0)$ and if $a_1 = a'_1, \ldots, a_{j-1} = a'_{j-1}$, $a_j < a'_j \cdot \gamma_1(\alpha') = [\ell']$, $\gamma_1(\alpha) = [\ell]$, $\ell' \geqslant j$ and we have to compare ℓ, ℓ' and j, we prove the strict inequality $(\alpha - \gamma_1(\alpha)) + [1] < (\alpha' - \gamma_1(\alpha')) + [1]$ directly if $\ell < j$ or $\ell = j < \ell'$ or $\ell = \ell' = j$, or $\ell > j$ and $\ell' > j$, or $\ell > j$, $\ell' = j$ and $a_j < a'_j - 1$.

The last case: $\ell > j$, $\ell' = j$ and $a_j = a'_j - 1$ implies:

$$\alpha = (\alpha' - [\ell']) + [\ell] \quad \text{since} \quad |\alpha| = |\alpha'| \quad \text{thus}$$

$$(\alpha - \gamma_1(\alpha)) + [1] = (\alpha' - \gamma_1(\alpha')) + [1] \quad \text{and} \quad \gamma_1(\alpha) + [2] < \gamma_1(\alpha') + [2].$$

The lemma, in the case $r = 1$, follows easily.

In the general case, we prove first:

$$\gamma_r(\alpha) = \gamma_1(\alpha) + \gamma_{r-1}(\alpha - \gamma_1(\alpha)).$$

Then, we suppose by induction the lemma proved for $r - 1$. We consider the set:

$$F_{s-1} = \{\alpha - \gamma \quad \text{where} \quad \alpha \in E_s, \quad |\gamma| = 1 \quad \text{and} \quad (\alpha - \gamma) > 0\}$$

$$\operatorname{Sup} F_{s-1} = \alpha_0 - \gamma_1(\alpha_0)$$

and

$$\operatorname{Sup} M(F_{s-1}, \mathfrak{G}, r-1) = (\alpha_0 - \gamma_1(\alpha_0) - \gamma_{r-1}(\alpha_0 - \gamma_1(\alpha_0)) + [1],$$

$$\gamma_{r-1}(\alpha_0 - \gamma_1(\alpha_0)) + [2])$$

$$= ((\alpha_0 - \gamma_r(\alpha_0)) + [1],\ \ \gamma_r(\alpha_0) - \gamma_1(\alpha_0) + [2]).$$

Thus for any α of E_s, γ such that $|\gamma| = r$, $(\alpha - \gamma) > 0$, i, j such that $[X_i, X_j] \neq 0$,

$$((\alpha - \gamma) + [i],\ (\gamma - \gamma_1(\gamma)) + [j]) =$$

$$= ((\alpha - \gamma_1(\gamma)) - (\gamma - \gamma_1(\gamma)) + [i],\ \ (\gamma - \gamma_1(\gamma)) + [j])$$

$$\leqslant ((\alpha_0 - \gamma_r(\alpha_0)) + [1],\ \ \gamma_r(\alpha_0) - \gamma_1(\alpha_0) + [2])$$

if $(\alpha - \gamma) + [i] = (\alpha_0 - \gamma_r(\alpha_0)) + [1]$ then i is 1.

Let us suppose in this case that

$$\gamma_r(\alpha_0) < \gamma$$

then:

$$(\gamma_r(\alpha_0) - \gamma_1(\alpha_0) + [1],\ \gamma_1(\alpha_0) + [2]) =$$

$$= (\gamma_r(\alpha_0) - \gamma_1(\gamma_r(\alpha_0)) + [1],\ \ \gamma_1(\gamma_r(\alpha_0)) + [2])$$

$$< ((\gamma - \gamma_1(\gamma)) + [1],\ \ \gamma_1(\gamma) + [2])$$

and:

$$(\gamma - \gamma_1(\gamma)) + [j] \leqslant (\gamma_r(\alpha_0) - \gamma_1(\alpha_0)) + [2] \quad \text{with} \quad j \geqslant 2.$$

The only possibility is:

$$\gamma_r(\alpha_0) - \gamma_1(\alpha_0) = \gamma - \gamma_1(\gamma),\ \ j = 2,\ \ \gamma_1(\alpha_0) < \gamma_1(\gamma)$$

but then

$$\alpha_0 = (\alpha_0 - \gamma_r(\alpha_0)) + (\gamma_r(\alpha_0) - \gamma_1(\alpha_0)) +$$
$$+ \gamma_1(\alpha_0) < (\alpha - \gamma) + (\gamma - \gamma_1(\gamma)) + \gamma_1(\gamma) = \alpha$$

which is impossible thus our lemma is proved. ■

5. THE FIRST COHOMOLOGICAL GROUP

Let us denote by $H^p(\mathfrak{G}, N)$ the p^{th} group of cohomology of the $\mathfrak{G}$-module N:

A p-cochain is a completely antisymmetric p-linear application c from $\mathfrak{G} \times \times \ldots \times \mathfrak{G}$ into N, the cohomology operator is:

$$(d\,c)(X^0, X^1, \ldots, X^p) = \sum_{i=0}^{p} (-1)^i \pi(X^i) c(X^0, \ldots, \hat{X}^i, \ldots, X^p)$$

$$- \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} c([X^i, X^j], X^0, \ldots, \hat{X}^i, \ldots, \hat{X}^j, \ldots, X^p), \quad (X^0, \ldots, X^p \in \mathfrak{G})$$

THEOREM 2. $([\mathfrak{G}, \mathfrak{G}] \neq 0): H^1_{\text{diff}}(N, \partial) \simeq H^1(\mathfrak{G}, N)$.
The isomorphism being defined by:

$$[C] \to [c] \quad \textit{where} \quad c(X) = C(x).$$

Let us remark that in the case $N = S(\mathfrak{G})$, the group $H^1(\mathfrak{G}, N)$ is the direct sum $\bigoplus_{m=0}^{\infty} H^1(\mathfrak{G}, S^m(\mathfrak{G}))$ where $S^m(\mathfrak{G})$ is the space of homogeneous polynomial with degree m, in particular if $\mathfrak{G}$ is semi simple, $H^1(\mathfrak{G}, S(\mathfrak{G}))$ vanishes.

Proof. Let us denote by $[x_i, x_j]$ the function $\{x_i, x_j\}$. Let C be a 1-cochain:

$$C = \sum_{|\alpha|>0} C_\alpha D^\alpha.$$

Then with our notations:

$$\partial C = \sum_{ij\alpha} [x_i, x_j]((D^{[j]}C_\alpha)D^{[i]} \cup D^\alpha + (D^{[i]}C_\alpha)D^\alpha \cup D^{[j]}$$

$$- C_\alpha \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|\geqslant 1, |\gamma|\geqslant 1}} \frac{\alpha!}{\beta!\gamma!} D^{\beta+[i]} \cup D^{\gamma+[j]})$$

$$- \sum_{ijk\alpha} C^k_{ij} C_\alpha \sum_{\substack{\beta+\gamma=\alpha-[k] \\ (\alpha-[k])>0}} \frac{(\alpha-[k])!}{\beta!\gamma!} D^{\beta+[i]} \cup D^{\gamma+[j]}.$$

If $s = \operatorname{Sup} |\alpha|$ is larger than 1, the biggest order of ∂C is $(s, 2)$ and the largest couple of multi indexes happening in

$$\partial C = \sum_{\alpha', \beta'} (\partial C)_{\alpha', \beta'} D^{\alpha'} \cup D^{\beta'}$$

is $(\alpha'_0, \beta'_0) = ((\alpha_0 - \gamma_1(\alpha_0)) + [1], \gamma_1(\alpha_0) + [2])$ where $\alpha_0 = \operatorname{Sup} \alpha$ and thanks to lemma 2:

$$(\partial C)_{\alpha'_0, \beta'_0} = -[x_1, x_2] C_{\alpha_0} \frac{\alpha_0!}{(\alpha_0 - \gamma_1(\alpha_0))!}.$$

This proves that $s = 1$, if C is a cocycle, and

$$C = \sum_{|\alpha|=1} C_\alpha D^\alpha$$

is 1-differential. Then we put:

$$c(X) = C(x) \qquad \forall x \in \mathfrak{G}.$$

c is a cocycle since:

$$(\mathrm{d}c)(X, Y) = (\partial C)(x, y) = 0$$

and c is a coboundary if and only if C is also a coboundary.

Finally, given a cocycle c in $Z^1(\mathfrak{G}, N)$, the 1-differential operator:

$$C = \sum_{|\alpha|=1} C_\alpha D^\alpha \quad \text{defined by } C(x) = c(X)$$

is a cocycle by construction: ∂C has only terms of order $(1, 1)$ thus it vanishes if and only if it vanishes on $\mathfrak{G} \times \mathfrak{G}$. The isomorphism of our theorem is proved. ∎

Remark. Our result is exactly the usual one: the differential 1-cocycles on symplectic manifolds are all 1-differential (cf. [3] for instance).

6. THE SECOND COHOMOLOGICAL GROUP

In the symplectic case, it happens a very important 2-cocycle, S^3_Γ ([1]): S^3_Γ is non exact and with 1-differential cohomology it generates the second cohomological group. Let us define here a similar cocycle.

Let us denote by σ the symmetrisation map from $S(\mathfrak{G})$ to the universal enveloping algebra $\mathcal{U}(\mathfrak{G})$ of $\mathfrak{G}$. $S(\mathfrak{G})$ is graded by the spaces S^r of homogeneous with degree r polynomial, thus $\mathcal{U}(\mathfrak{G})$ admits a decomposition:

$$\mathcal{U}(\mathfrak{G}) = \bigoplus_{r=0}^{\infty} \sigma(S^r).$$

Let us denote by u_r the r^{th}-component of u of $S(\mathfrak{G})$ in this decomposition. Finally if $\circ$ is the product in $\mathcal{U}(\mathfrak{G})$, we put:

$$S^3(u, v) = \sigma^{-1}(\sigma(u) \circ \sigma(v))_{r+s-3} \quad \text{if} \quad u \in S^r, \quad v \in S^s .$$

PROPOSITION 2. ([1]) ($[\mathfrak{G}, \mathfrak{G}] \neq 0$). *$S^3$ is a differential cocycle on $S(\mathfrak{G})$ (thus on $C^\infty(\mathfrak{G}^*)$). S^3 is non exact and if:*

$$\alpha_0 = [1] + [1] + [1], \quad \beta_0 = [2] + [2] + [2],$$

the largest (α, β) *happening in* $S^3 = \sum_{\alpha,\beta} S^3_{\alpha,\beta} D^\alpha \cup D^\alpha$ *is* (α_0, β_0), *moreover:*

$$S^3_{\alpha_0,\beta_0} = [x_1, x_2]^3.$$

Proof. S. Gutt proved in [11] that we can define bidifferential operators C_2, $C_3, \ldots$ on $\mathfrak{G}^*$ by induction such that:

$$u * v = uv + \nu\{u, v\} + \nu^2 C_2(u, v) + \nu^3 C_3(u, v) \ \ldots$$

is a * product, i.e. is associative, the C_i vanish on constants and $C_i(u, v) = (-i)^i$ $C_i(v, u)$.

Computing explicitely the C_i, she proved:

$$C_3(u, v) = S^3(u, v).$$

Thus S^3 is a differential cocycle. Moreover, the C_i are defined by symmetrization or anti-symmetrization of an explicit solution of the associativity equations:

$$\tilde{\partial}\tilde{C}_i = E_i$$

where

$$(\tilde{\partial}\tilde{C})(u, v, w) = u\tilde{C}(v, w) - \tilde{C}(uv, w) + \tilde{C}(u, vw) - \tilde{C}(u, v) \cdot w.$$

The first equation is:

$$(\tilde{\partial}\tilde{C}_2)(u, v, w) = \{v, \{u, w\}\} = \sum_{ii'jj'kk'} C^k_{ij} C^{k'}_{i'j'} x_{k'} [\delta_{j'k} D^{[i]} \cup D^{[i']} \cup D^{[j]}$$

$$+ x_k D^{[i]+[j']} \cup D^{[i']} \cup D^{[j]} + x_k D^{[i]} \cup D^{[i']} \cup D^{[j]+[j']}] \quad (u, v, w).$$

Thus C_2 has the form:

$$\tilde{C}_2 = \Sigma\, \tilde{C}_{\alpha\beta}\, D^\alpha \cup D^\beta$$

where:

$$(|\alpha|, |\beta|) \leqslant (2, 2) \qquad \text{(the orders (3, 1), (1, 3) vanish).}$$

And if:

$$(\alpha_1, \beta_1) = \operatorname{Sup}\{(\alpha, \beta) \quad \text{s.t.} \quad \tilde{C}_{\alpha,\beta} \neq 0\},$$

we see:

$$\alpha_1 \leqslant [1] + [1].$$

If $\widetilde{C}_{[1]+[1],\beta} \neq 0$ and $|\beta| = 2$, then $1 \notin \operatorname{Supp} \beta$ thus $(\alpha_1, \beta_1) \leqslant ([1]+[1], [2]+[2])$.

But the term

$$\widetilde{C}_{[1]+[1],[2]+[2]} \quad \text{is} \quad -2[x_1, x_2]^2$$

(see the explicit solution in [11], lemma 1), it is the same for C_2.

Now the same proof is available pour C_3

$$(\widetilde{\delta}\widetilde{C}_3)(u, v, w) = - \ u, C_2(v, w) \ + C_2(\{u, v\}, w) - \\ - C_2(u, \{v, w\}) + \{C_2(u, v), w\}.$$

If we put:

$$\widetilde{C}_3 = \Sigma \ C'_{\alpha,\beta} D^{\alpha} \cup D^{\beta},$$

then $\operatorname{Sup}\{|\alpha| + |\beta| / C'_{\alpha,\beta} \neq 0\} = 6$.

There is no terms with order (4, 1, 1) in E_3, thus no term with order (5, 1) or (4, 2) in $\widetilde{C}_3$ ([11], lemma 1) and if

$$(\alpha_0, \beta_0) = \operatorname{Sup}\{(\alpha, \beta)/C'_{\alpha,\beta} \neq 0\}, \quad \alpha_0 \leqslant [1]+[1]+[1].$$

If $E_3 = \Sigma \ E_{\alpha\beta\gamma} D^{\alpha} \cup D^{\gamma} \cup D^{\gamma}$, we verify case by case that:

$$E_{[1]+[1]+[1],[1]+[i],[j]} = E_{[1]+[1]+[1],[i]+[j],[1]} = \\ E_{[1]+[1],[1]+[i],[1]+[j]} = E_{[1]+[1],[1]+[1],[i]+[j]} = 0,$$

then if,

$$C'_{[1]+[1]+[1],\beta} \neq 0 \quad \text{and} \quad |\beta| = 3, \ 1 \notin \operatorname{Supp} \beta$$

and $(\alpha_0, \beta_0) \leqslant ([1]+[1]+[1], [2]+[2]+[2])$.

Finally we compute the term

$$C'_{[1]+[1]+[1],[2]+[2]+[2]} = [x_1, x_2]^3 = - C'_{[2]+[2]+[2],[1]+[1]+[1]} \neq 0$$

which proves our proposition since if

$$B = \Sigma \ B_{\alpha} D^{\alpha} \ \text{ with } \operatorname{Sup}\{\alpha \text{ s.t. } B_{\alpha} \neq 0\} = \alpha_1,$$

$$\partial B = \Sigma(\partial B)_{\alpha,\beta} D^{\alpha} \cup D^{\beta} \quad \text{with} \quad \operatorname{Sup}\{(\alpha, \beta) \text{ s.t. } (\partial B)_{\alpha,\beta} \neq 0\} = \\ ((\alpha_1 - \gamma_1(\alpha_1) + [1]), \ \gamma_1(\alpha_1) + [2])$$

with order distinct of (3, 3). ■

Remarks. 1) Let $\lambda \in I$ and $\lambda \neq 0$, then the operator λS^3 is, as S^3, a differential non exact 2-cocycle.

2) By the definition of $*$, we have: for all λ in I:

$$\lambda_* u = u * \lambda \quad u \in N, \quad \lambda \in I \text{ thus } S^3(u, \lambda) = 0$$

S^3 is tangential in the sense of [14].

Let us now consider a 2-cocycle:

$$C = \Sigma\, C_{\alpha,\beta} D^\alpha \cup D^\beta$$

if:

$$F(C) = \ (\alpha, \beta) \ \text{ s.t. } \ C_{\alpha,\beta} \neq 0 \ , \ \ (\alpha_0, \beta_0) = \text{Sup}\, F(C),$$

we shall compute ∂C and, following the method of S. Gutt in [3], study first $(|\alpha_0|, |\beta_0|)$. We put $(r, s) = (|\alpha_0|, |\beta_0|)$.

PROPOSITION 3. ($[\mathfrak{G}, \mathfrak{G}] \neq 0$). *The only possibilities for* $(r, s) = (|\alpha_0|, |\beta_0|)$ *are* $(r, 2)$ *with* $r \geqslant 2$, $(3, 3)$ *or* $(1, 1)$.

Proof. We have:

$$\partial C = \sum_{\alpha\beta ij} [x_i, x_j] D^{[j]} C_{\alpha\beta} (D^{[i]} \cup D^\alpha \cup D^\beta -$$

$$- D^\alpha \cup D^{[i]} \cup D^\beta + D^\alpha \cup D^\beta \cup D^{[i]})$$

$$+ \sum_{\alpha\beta ij} [x_i, x_j] C_{\alpha\beta} (D^{[i]} \cup D^{\alpha+[j]} \cup D^\beta +$$

$$+ D^{[i]} \cup D^\alpha \cup D^{\beta+[j]} - D^{\alpha+[j]} \cup D^{[i]} \cup D^\beta$$

$$- D^\alpha \cup D^{[i]} \cup D^{\beta+[j]} + D^{\alpha+[j]} \cup D^\beta \cup D^{[i]} +$$

$$+ D^\alpha \cup D^{\beta+[j]} \cup D^{[i]})$$

$$- \sum_{\alpha\beta ijk} C_{\alpha\beta} C_{ij}^k \sum_{\gamma+\delta=\alpha-[k]} \frac{(\alpha-[k])!}{\gamma!\delta!} (D^{\gamma+[i]} \cup D^{\delta+[j]} \cup D^\beta -$$

$$- D^{\gamma+[i]} \cup D^\beta \cup D^{\delta+[j]} + D^\beta \cup D^{\gamma+[i]} \cup D^{\delta+[j]})$$

$$- \sum_{\alpha\beta ij} C_{\alpha\beta} [x_i, x_j] \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} (D^{\gamma+[i]} \cup D^{\delta+[j]} \cup D^\beta -$$

$$- D^{\gamma+[i]} \cup D^\beta \cup D^{\delta+[j]} + D^\beta \cup D^{\gamma+[i]} \cup D^{\delta+[j]}).$$

After simplifications, we can write:

$$\partial C = \sum_{\alpha\beta ij} [x_i, x_j] D^{[j]} C_{\alpha\beta} (D^{[i]} \cup D^\alpha \cup D^\beta -$$

$$- D^\alpha \cup D^{[i]} \cup D^\beta + D^\alpha \cup D^\beta \cup D^{[i]})$$

$$- \sum_{\alpha\beta ijk} C_{\alpha\beta} C_{ij}^k \sum_{\gamma+\delta=\alpha-[k]} \frac{(\alpha-[k])!}{\gamma!\delta!} (D^{\gamma+[i]} \cup D^{\delta+[j]} \cup D^\beta -$$

$$- D^{\gamma+[i]} \cup D^\beta \cup D^{\delta+[j]} + D^\beta \cup D^{\gamma+[i]} \cup D^{\delta+[j]})$$

$$- \sum_{\alpha\beta ij} [x_i, x_j] C_{\alpha\beta} \sum_{\substack{\gamma+\delta=\alpha \\ |\gamma|\geqslant 1, |\delta|\geqslant 1}} \frac{\alpha!}{\gamma!\delta!} (D^{\gamma+[i]} \cup D^{\delta+[j]} \cup D^\beta -$$

$$- D^{\gamma+[i]} \cup D^\beta \cup D^{\delta+[j]} + D^\beta \cup D^{\gamma+[i]} \cup D^{\delta+[j]})$$

The orders of these terms are respectively:

$$(1, |\alpha|, |\beta|), \quad (|\alpha|, 1, |\beta|), \quad (|\alpha|, |\beta|, 1)$$
$$(j+1, |\alpha|-j, |\beta|), \quad (j+1, |\beta|, |\alpha|-j),$$
$$(|\beta|, j+1, |\alpha|-j) \qquad \text{if } |\alpha|>1, \ 0\leqslant j\leqslant |\alpha|-1$$
$$(j+1, |\alpha|-j+1, |\beta|), \quad (j+1, |\beta|, |\alpha|-j+1),$$
$$(|\beta|, j+1, |\alpha|-j+1) \quad \text{if } |\alpha|>1, \ 1\leqslant j\leqslant |\alpha|-1$$

Let us denote by $(\theta_{\ell m}(\alpha, \beta, j))$ $1 \leqslant \ell \leqslant 3$, $1 \leqslant m \leqslant 3$ that matrix of orders, for instance:

$$\theta_{23}(\alpha, \beta, j) = (|\beta|, j+1, |\alpha|-j).$$

First case: $r > s > 2$ is impossible.

If $r \geqslant s > 2$, the maximal order in ∂C is $(r, s, 2)$ obtained with $\theta_{32}(\alpha, \beta, |\alpha_0|-1)$ and $\theta_{33}(\beta, \alpha, |\alpha_0|-1)$ for $(|\alpha|, |\beta|) = (r, s)$. Thus in ∂C the corresponding term must vanish:

$$0 = \sum_{(|\gamma|, |\delta|, |\epsilon|)=(r,s,2)} (\partial C)_{\gamma,\delta,\epsilon} D^\gamma \cup D^\delta \cup D^\epsilon$$

$$= \sum_{i,j,(|\alpha|,|\beta|)=(r,s)} [x_i, x_j] C_{\alpha,\beta} \left[\sum_{\substack{|\gamma|=1 \\ (\alpha-\gamma)>0}} \frac{\alpha!}{(\alpha-\gamma)!} D^{\alpha-\gamma+[i]} \cup D^\beta \cup D^{\gamma+[j]} \right.$$

$$+ \sum_{\substack{|\gamma|=1 \\ (\beta-\gamma)>0}} \frac{\beta!}{(\beta-\gamma)!} D^{\alpha} \cup D^{\beta-\gamma+[i]} \cup D^{\gamma+[j]} \Bigg]$$

(C is antisymmetric). If $\gamma_1(\alpha_0) \neq [1]$, we find:

$$(\partial C)_{(\alpha_0-\gamma_1(\alpha_0)+[1],\beta_0,\gamma_1(\alpha_0)+[2])} =$$

$$= \frac{\alpha_0!}{(\alpha_0-\gamma_1(\alpha_0))!} [x_1, x_2] C_{\alpha_0,\beta_0} = 0.$$

This is impossible, thus $\alpha_0 = [1] + \ldots + [1]$. Now:

$$(\partial C)_{(\alpha_0,\beta_0,[1]+[2])} = [x_1, x_2] C_{\alpha_0,\beta_0} r + \frac{\beta_0!}{(\beta_0-[1])!} [x_1, x_2] C_{\alpha_0,\beta_0}$$

$$+ \frac{\beta_0!}{(\beta_0-[2])!} [x_2, x_1] C_{\alpha_0,\beta_0}.$$

The second and third terms happening only if 1 (resp. 2) belongs to Supp β_0.

Let us write $\beta_0 = (b_1, b_2, \ldots, b_n)$ we have the equations:

$$r + b_1 - b_2 = 0 \qquad s = b_1 + b_2 + \ldots + b_n \leqslant r.$$

The only possibility is:

$$r = s, \qquad \beta_0 = [2] + \ldots + [2].$$

Second case: $r = s > 3$ is impossible.

Let us suppose that all the terms with order $(r, r, 2)$ in ∂C vanish, the maximal order is thus $(r, r-1, 3)$, it is happening with:

$$\theta_{33}(\alpha, \beta, |\alpha| - 2) \qquad \text{for} \qquad |\alpha| = |\beta| = r.$$

We find:

$$\sum_{|\alpha|=|\beta|=r, ij} [x_i, x_j] C_{\alpha\beta} \sum_{\substack{|\gamma|=2 \\ (\beta-\gamma)>0}} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^{\alpha} \cup D^{(\beta-\gamma)+[i]} \cup D^{\gamma+[j]} = 0$$

with lemma 2, we have:

$$(\partial C)_{\alpha_0,\beta_0-\gamma_2(\beta_0)+[1],\gamma_2(\beta_0)+[2]} =$$

$$= [x_1, x_2] C_{\alpha_0,\beta_0} \frac{\beta_0!}{\gamma_2(\beta_0)!(\beta_0-\gamma_2(\beta_0))!} = 0$$

and it is impossible.

Third case: $r > s = 1$ is impossible.

The only terme with order $(r, 2, 1)$ in ∂C comes with $\theta_{31}(\alpha, \beta, |\alpha| - 1)$ for $(|\alpha|, |\beta|) = (r, 1)$ it is:

$$0 = \sum_{ij(|\alpha|,|\beta|)=(r,1)} [x_i, x_j] C_{\alpha,\beta} \sum_{\substack{|\gamma|=1 \\ (\alpha-\gamma)>0}} \frac{\alpha!}{(\alpha-\gamma)!} D^{\alpha-\gamma+[i]} \cup D^{\gamma+[j]} \cup D^{\beta}.$$

Thus:

$$(\partial C)_{(\alpha_0-\gamma_1(\alpha_0)+[1],\gamma_1(\alpha_0)+[2],\beta_0)} = $$
$$= \frac{\alpha_0!}{(\alpha_0-\gamma_1(\alpha_0))!} [x_1, x_2] C_{\alpha_0,\beta_0} = 0$$

that is impossible. ∎

PROPOSITION 4. $([\mathfrak{G}, \mathfrak{G}] \neq 0)$. *If $(r, s) = (3, 3)$ then*
a) if $\mathfrak{G} \neq h_r \oplus \mathbb{R}^m$, there exists $\lambda \in I$ such that
$C - \lambda S^3$ has no term with order (3, 3);
b) if $\mathfrak{G} = h_r \oplus \mathbb{R}^m$, there exists $\lambda \in I$ such that
$C - \dfrac{\lambda}{[x_1, x_2]^3} S^3$ has no term with order (3, 3).

Proof. We saw in the preceding proof that if $(r, s) = (3, 3)$ then

$$\alpha_0 = [1] + [1] + [1], \quad \beta_0 = [2] + [2] + [2].$$

Let us determine $(\partial C)_{\alpha_0,\beta_0,[1]+[k]}$; it is a term of order (3, 3, 2) which happens only with $\theta_{32}(\alpha, \beta, |\alpha| - 1)$ and $\theta_{33}(\alpha, \beta, |\alpha| - 1)$ with $|\alpha| = |\beta| = 3$, we find:

$$0 = (\partial C)_{\alpha_0,\beta_0,[1]+[k]} = 3[x_1, x_k] C_{\alpha_0,\beta_0} -$$
$$- \frac{(\beta_0 - [2] + [k])!}{(\beta_0 - [2])!} [x_1, x_2] C_{\alpha_0,(\beta_0-[2])+[k]}.$$

This equation is non trivial only for $k > 2$ and it is:

$$3[x_1, x_k] C_{\alpha_0,\beta_0} = [x_1, x_2] C_{\alpha_0,\beta_0-[2]+[k]} \qquad (k > 2).$$

Similarly, we find if $k > 1$:

$$0 = (\partial C)_{\alpha_0,\beta_0,[2]+[k]} = [x_1, x_2] C_{\alpha_0-[1]+[k],\beta_0} + [x_1, x_k] C_{\alpha_0-[1]+[2],\beta_0}$$
$$- 3[x_2, x_k] C_{\alpha_0,\beta_0}.$$

If $k = 2$ this relation imposes $C_{\alpha_0 - [1] + [2], \beta_0} = 0$ thus:

$$[x_1, x_2] C_{\alpha_0 - [1] + [k], \beta_0} = 3[x_1, x_k] C_{\alpha_0, \beta_0} \qquad \forall k > 2.$$

Let us consider now $(\partial C)_{\alpha_0, \beta_0, [i]}$ its order is (3, 3, 1), it comes from terms with order $\theta_{13}(\alpha, \beta)$ $|\alpha| = |\beta| = 3$, $\theta_{22}(\alpha, \beta, |\alpha| - 1)$ or $\theta_{23}(\alpha, \beta, |\alpha| - 1)$ with $|\alpha|, |\beta| = 3$. We find:

$$0 = (\partial C)_{\alpha_0, \beta_0, [i]} = \sum_j [x_i, x_j] D^{[j]} C_{\alpha_0, \beta_0} + \sum_k C^k_{1i} C_{\alpha_0 - [1] + [k], \beta_0}$$

$$- \sum_k C^k_{2i} C_{\beta_0 - [2] + [k], \alpha_0}.$$

The sums are only on $k > 2$ and on the open set $[x_1, x_2] \neq 0$ of $\mathfrak{G}^*$, C_{α_0, β_0} satisfies the differential equation:

$$\{x_i, C_{\alpha_0, \beta_0}\} + 3 \sum_{k > 2} C^k_{1i} \frac{[x_2, x_k]}{[x_2, x_1]} C_{\alpha_0, \beta_0} +$$

$$+ 3 \sum_{k > 2} C^k_{2i} \frac{[x_1, x_k]}{[x_1, x_2]} C_{\alpha_0, \beta_0} = 0$$

or:

$$\{x_i, C_{\alpha_0, \beta_0}\} - \frac{3}{[x_1, x_2]} \{x_i, [x_1, x_2]\} C_{\alpha_0, \beta_0} = 0 \qquad \forall i \qquad (*).$$

On the open set $[x_1, x_2] \neq 0$, we write $C_{\alpha_0, \beta_0} = f([x_1, x_2])^3$ and (*) becomes $\{x_i, f\} = 0$ $\forall i$.

f is «rationnal» and $C - fS^3$ is a cocycle without term in $D^{\alpha_0} \cup D^{\beta_0}$, its order is strictly less than (3, 3).

First case: $\dim[\mathfrak{G}, \mathfrak{G}] > 1$.

We can consider the new basis $(e'_1 \ldots e'_n) = (e_1, e_3, e_2, e_4 \ldots e_n)$.

The preceding argument give us:

$$S^3_{\alpha_0, [3] + [3] + [3]} = ([x_1, x_3])^3.$$

On the open set $[x_1, x_2] \neq 0$, $[x_1, x_3] \neq 0$,

$$C_{\alpha_0, \beta_0} = f[x_1, x_2]^3, \quad C_{\alpha_0, [3] + [3] + [3]} = f[x_1, x_3]^3.$$

Thus everywhere:

$$C_{\alpha_0, \beta_0}[x_1, x_3]^3 = C_{\alpha_0, [3] + [3] + [3]}[x_1, x_2]^3$$

and $C_{\alpha_0,\beta_0} \dfrac{1}{[x_1, x_2]^3} = f$ is an element λ of I.

Second case: $\dim [\mathfrak{G}, \mathfrak{G}] = 1$, $[\mathfrak{G}, \mathfrak{G}]$ *non central:*

For $i = 1$ (*) gives us:

$$\{x_1, C_{\alpha_0,\beta_0}\} - 3C_{\alpha_0,\beta_0} = 0 = x_2 \partial_2 C_{\alpha_0,\beta_0} - 3C_{\alpha_0,\beta_0}.$$

Thus $C_{\alpha_0,\beta_0} = f.\, x_2^3$ where f is in N, thus f belongs to I.

Third case: $\mathfrak{G} = h_r \oplus \mathbb{R}^m$ $r > 0$.

In this case $\dfrac{1}{[x_1, x_2]^3} S^3$ is a 2-cocycle, in fact the * product of [11] which allows us to compute S^3 is exactly the «moyal product» on $\mathfrak{G}^*$ ([14]).

We define the 2-tensor Λ on $\mathfrak{G}^*$ by the relation:

$$i(\Lambda)(x_i, x_j)_\xi = \langle \xi, [x_i, x_j] \rangle = \Lambda^{ij},$$

thus:

$$S^3 = \frac{1}{3!} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \Lambda^{i_3 j_3} D^{[i_1] + [i_2] + [i_3]} \cup D^{[j_1] + [j_2] + [j_3]}.$$

The non vanishing Λ^{ij} are all $\pm [x_1, x_2]$ thus $\dfrac{1}{[x_1, x_2]^3} S^3$ is a cochain on N, it is a 2-cocycle since $[x_1, x_2]$ is invariant then:

$$f = \frac{\lambda}{[x_1, x_2]^3}, \quad \lambda \in I \qquad \text{Q.E.D.}$$

■

PROPOSITION 5. ($[\mathfrak{G}, \mathfrak{G}] \neq 0$): *If* $(r, s) = (r, 2)$, *then* $1 \in \operatorname{Supp} \alpha_0$, $2 \in \operatorname{Supp} \beta_0$ *and* $\gamma_1(\beta_0) \leqslant \gamma_1(\alpha_0)$.

Proof. Let us consider the term with order $(r, 2, 2)$ in ∂C they come from terms with order:

$$\theta_{31}(\alpha, \beta, r-1), \qquad \theta_{32}(\alpha, \beta, r-1) \qquad \text{and}$$

$$\theta_{33}(\beta, \alpha, 1) \quad \text{with} \quad (|\alpha|, |\beta|) = (r, 2).$$

We find:

$$-\sum_{\substack{(|\alpha|, |\beta|) = (r,2) \\ i,j}} [x_i, x_j] C_{\alpha\beta} \,\cdot$$

$$(*) \qquad \cdot\left[\sum_{\substack{|\gamma|=1 \\ (\alpha-\gamma)>0}} \frac{\alpha!}{(\alpha-\gamma)!} (D^{\alpha-\gamma+[i]} \cup D^{\gamma+[j]} \cup D^{\beta} - D^{\alpha-\gamma+[i]} \cup D^{\beta} \cup D^{\gamma+[j]}) - \sum_{\substack{|\gamma|=1 \\ (\beta-\gamma)>0}} \frac{\beta!}{(\beta-\gamma)!} D^{\alpha} \cup D^{\beta-\gamma+[i]} \cup D^{\gamma+[j]} \right] = 0.$$

Let us suppose $1 \notin \operatorname{Supp} \alpha_0$ and put: $\beta_0 = [i_0] + [j_0]$ with $i_0 \leqslant j_0$. We have: $1 \notin \operatorname{Supp} \alpha$ for all α in (*) thus:

$$\begin{aligned} 0 &= (\partial C)_{\alpha_0-\gamma_1(\alpha_0)+[1],\gamma_1(\alpha_0)+[2],\beta_0} = \\ &= -[x_1, x_2] \frac{\alpha_0!}{(\alpha_0-\gamma_1(\alpha_0))!} C_{\alpha_0,\beta_0} + \\ &+ [x_1, x_{j_0}] \frac{(\alpha_0-\gamma_1(\alpha_0)+[i_0])!}{(\alpha_0-\gamma_1(\alpha_0))!} C_{\alpha_0-\gamma_1(\alpha_0)+[i_0],\gamma_1(\alpha_0)+[2]} + \\ &+ [x_1, x_{i_0}] \frac{(\alpha_0-\gamma_1(\alpha_0)+[j_0])!}{(\alpha_0-\gamma_1(\alpha_0))!} C_{\alpha_0-\gamma_1(\alpha_0)+[j_0],\gamma_1(\alpha_0)+[2]}. \end{aligned}$$

This proves that $C_{\alpha_0-\gamma_1(\alpha_0)+[i],\gamma_1(\alpha_0)+[2]} \neq 0$ for $i = i_0$ or j_0. Let us suppose $\gamma_1(\alpha_0) < [2]$ strictly, we find

$$\begin{aligned} 0 &= (\partial C)_{\alpha_0-\gamma_1(\alpha_0)+[i],[1]+[2],[2]+\gamma_1(\alpha_0)} \\ &= \sum_{k \leqslant i} [x_k, x_2] C_{\alpha_0-[k]+[i],[1]+[2]} \frac{(\alpha_0-[k]+[i])!}{(\alpha_0-\gamma_1(\alpha_0))!} + \\ &+ [x_1, x_2] C_{\alpha_0-\gamma_1(\alpha_0)+[i],\gamma_1(\alpha_0)+[2]} + \\ &+ 2[x_1, x_{\gamma_1(\alpha_0)}] C_{\alpha_0-\gamma_1(\alpha_0)+[i],[2]+[2]} + \\ &+ [x_2, x_{\gamma_1(\alpha_0)}] C_{\alpha_0-\gamma_1(\alpha_0)+[i],[1]+[2]}. \end{aligned}$$

If $\gamma_1(\alpha_0) = [2]$, $\alpha_0 = [2] + \ldots + [2]$, let us consider:

$$\begin{aligned} 0 &= (\partial C)_{\alpha_0-[2]+[1],[2]+[2],\beta_0} = -[x_1, x_2] r\, C_{\alpha_0,\beta_0} + \\ &+ [x_1, x_{j_0}] \frac{(\alpha_0-[2]+[i_0])!}{(\alpha_0-[2])!} C_{\alpha_0-[2]+[i_0],[2]+[2]} + \\ &+ [x_1, x_{i_0}] \frac{(\alpha_0-[2]+[j_0])!}{(\alpha_0-[2])!} C_{\alpha_0-[2]+[j_0],[2]+[2]}. \end{aligned}$$

Then if $1 \notin \operatorname{Supp} \alpha_0$, there exists α with $|\alpha| = r$ and $C_{\alpha,[1]+[2]} \neq 0$ or α with $|\alpha| = r$ and $C_{\alpha,[2]+[2]} \neq 0$. If $C_{\alpha,[2]+[2]} \neq 0$, we consider:

$$0 = (\partial C)_{\alpha,[2]+[1],[2]+[1]} =$$
$$= 2[x_2, x_1]C_{\alpha,[2]+[2]} + 2[x_1, x_2]C_{\alpha,[1]+[1]}.$$

If $C_{\alpha,[1]+[2]} \neq 0$, we consider:

$$0 = (\partial C)_{\alpha,[1]+[1],[2]+[2]} =$$
$$= -\sum_i [x_i, x_2] \frac{(\alpha - [i] + [2])!}{(\alpha - [i])!} C_{\alpha-[i]+[2],[1]+[1]} +$$
$$+ [x_1, x_2]C_{\alpha,[1]+[2]}.$$

In each case, if $1 \notin \operatorname{Supp} \alpha_0$, the set:

$$G = \{\alpha \text{ s.t. } |\alpha| = r \text{ and } C_{\alpha,[1]+[1]} \neq 0\}$$

is non void, let $\alpha_1 = \operatorname{Sup} G$. We find:

$$0 = (\partial C)_{\alpha_1-\gamma_1(\alpha_1)+[1],\gamma_1(\alpha_1)+[2],[1]+[1]} =$$
$$= -\frac{\alpha_1!}{(\alpha_1 - \gamma_1(\alpha_1))!} [x_1, x_2]C_{\alpha_1,[1]+[1]}$$

which is impossible, $1 \in \operatorname{Supp} \alpha_0$.

Let us suppose now $\gamma_1(\alpha_0) < [1]$ strictly, then:

$$0 = (\partial C)_{(\alpha_0-\gamma_1(\alpha_0))+[1],\gamma_1(\alpha_0)+[2],\beta_0} =$$
$$= -\frac{\alpha_0!}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_2]C_{\alpha_0,\beta_0} +$$
$$+ \frac{(\alpha_0 - \gamma_1(\alpha_0) + j_0)!}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_{i_0}]C_{\alpha_0-\gamma_1(\alpha_0)+[j_0],\gamma_1(\alpha_0)+[2]} +$$
$$+ \frac{(\alpha_0 - \gamma_1(\alpha_0) + i_0)!}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_{j_0}]C_{\alpha_0-\gamma_1(\alpha_0)+[i_0],\gamma_1(\alpha_0)+[2]}.$$

If $i_0 = 1$, $\alpha_0 - \gamma_1(\alpha_0) + [1]$ is larger than α_0 thus we find

$$0 = -\frac{\alpha_0!}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_2]C_{\alpha_0,\beta_0}$$

which is impossible. Now $i_0 > 1$. If

$$C_{\alpha_0 - \gamma_1(\alpha_0) + [i_0], \gamma_1(\alpha_0) + [2]} = 0,$$

we put $i = j_0$, if it is not the case, $i = i_0$. Then $[i] \leqslant \gamma_1(\alpha_0)$ and:

$$0 = (\partial C)_{\alpha_0 - \gamma_1(\alpha_0) + [1], [i] + [2], \gamma_1(\alpha_0) + [2]} =$$

$$= - \frac{(\alpha_0 - \gamma_1(\alpha_0) + [i])!}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_2] C_{\alpha_0 - \gamma_1(\alpha_0) + [i], \gamma_1(\alpha_0) + [2]} +$$

$$+ \frac{\alpha_0 !}{(\alpha_0 - \gamma_1(\alpha_0))!} [x_1, x_2] C_{\alpha_0, [2] + [i]}.$$

This proves that $C_{\alpha_0, [2] + [i]} \neq 0$, $[2] + [i] \leqslant \beta_0$, $1 \notin \operatorname{Supp} \beta_0$ then $2 \in \operatorname{Supp} \beta_0$, $\beta_0 = [2] + \gamma_1(\beta_0)$ and $\gamma_1(\beta_0) \leqslant \gamma_1(\alpha_0)$.

Let us suppose now $\gamma_1(\alpha_0) = [1]$ i.e. $\alpha_0 = [1] + \ldots + [1]$. Then:

$$0 = (\partial C)_{\alpha_0, [1] + [2], \beta_0} = - r [x_1, x_2] C_{\alpha_0, \beta_0} +$$

$$+ (r-1)[x_1, x_{i_0}] C_{\alpha_0 - [1] + [j_0], [1] + [2]} +$$

$$+ (r-1)[x_1, x_{j_0}] C_{\alpha_0 - [1] + [i_0], [1] + [2]} +$$

$$+ \frac{\beta_0 !}{(\beta_0 - [i_0])!} [x_1, x_{i_0}] C_{\alpha_0, [2] + [j_0]} +$$

$$+ \frac{\beta_0 !}{(\beta_0 - [j_0])!} [x_1, x_{j_0}] C_{\alpha_0, [2] + [i_0]} +$$

$$+ \frac{\beta_0 !}{(\beta_0 - [i_0])!} [x_2, x_{i_0}] C_{\alpha_0, [1] + [j_0]} +$$

$$+ \frac{\beta_0 !}{(\beta_0 - [j_0])!} [x_2, x_{j_0}] C_{\alpha_0, [1] + [i_0]}.$$

Here some terms can be not present, for instance if $i_0 = j_0 = 1$, we have only

$$0 = - (r+2)[x_1, x_2] C_{\alpha_0, \beta_0}$$

which is impossible, if $1 = i_0$ and $2 < j_0$, we obtain:

$$0 = - (r+1)[x_1, x_2] C_{\alpha_0, \beta_0}$$

which is also impossible, finally if $2 < i_0 \leqslant j_0$, we obtain:

$$0 = - r [x_1, x_2] C_{\alpha_0, \beta_0} + r [x_1, x_{i_0}] C_{\alpha_0 - [1] + [j_0], [1] + [2]} +$$

$$+ r[x_1, x_{j_0}]C_{\alpha_0-[1]+[i_0],[1]+[2]}$$

Now if $C_{\alpha_0-[1]+[i],[1]+[2]} \neq 0$, we see since $\beta_0 < [1] + [i]$, $[2] + [i]$ and $[2] + [2]$:

$$0 = (\partial C)_{\alpha_0,[2]+[i],[1]+[2]} = -(r-1)[x_1, x_2]C_{\alpha_0-[1]+[i],[1]+[2]} - [x_1, x_i]C_{\alpha_0-[1]+[2],[1]+[2]}.$$

And:

$$0 = (\partial C)_{\alpha_0,[2]+[2],[1]+[2]} = -(r-1)[x_1, x_2]C_{\alpha_0-[1]+[2],[1]+[2]}$$

which is impossible. The only remaining cases are: $\beta_0 = [1] + [2]$ or $\beta_0 = [2] + \gamma_1(\beta_0)$. Our proposition is proved. ■

PROPOSITION 6. ($[\mathfrak{G}, \mathfrak{G}] \neq 0$). *If* $\mathfrak{G} \neq h_r \oplus \mathbb{R}^m$ *and* $(r, s) = (r, 2)$, *then:*

$$C_{\alpha_0,\beta_0} = [x_1, x_2] \cdot f \quad where \quad f \in N.$$

Proof. We keep our preceding notations.

First case: $\dim [\mathfrak{G}, \mathfrak{G}] > 1$:

Let us suppose first $\gamma_1(\alpha_0) < [1]$ then $\beta_0 = [2] + \gamma_1(\beta_0)$ and $\gamma_1(\beta_0) \leqslant \gamma_1(\alpha_0)$. We put: $[a] = \gamma_1(\alpha_0)$, $[b] = \gamma_1(\beta_0)$. We have:

$$0 = (\partial C)_{\alpha_0-[a]+[1],[a]+[a],\beta_0} = -\frac{\alpha_0!}{(\alpha_1-[1])!}[x_1, x_a]C_{\alpha_0,\beta_0} + \frac{(\alpha_0-[a]+[1])!}{(\alpha_0-[a])!}[x_1, x_b]C_{\alpha_0,[a]+[a]} + \frac{(\alpha_0-[a]+[b])!}{(\alpha_0-[a]+[b]-[1])!}[x_1, x_2]C_{\alpha_0-[a]+[b],[a]+[a]}.$$

The second term happening only if $a = 2 = b$. Thus if it is not the case, our proposition is proved from lemma 1. If $a = b = 2$, we consider:

$$0 = (\partial C)_{\alpha_0-[2]+[1],[2]+[3],[2]+[2]} = -\frac{\alpha_0!}{(\alpha_0-[1])!}[x_1, x_3]C_{\alpha_0,\beta_0}$$

$$- \frac{(\alpha_0 - [2] + [3])!}{(\alpha_0 - [2] - [1] + [3])!} [x_1, x_2] C_{(\alpha_0 - [2] + [3]), \beta_0} +$$

$$+ \frac{\alpha_0 !}{(\alpha_0 - [1])!} [x_1, x_2] C_{\alpha_0, [2] + [3]}.$$

And the conclusion follows from lemma 1.

Let us suppose now $\gamma_1(\alpha_0) = [1]$, then $\beta_0 = [2] + \gamma_1(\beta_0)$ or $[2] + [2]$ or $[1] + [2]$. If $\beta_0 = [2] + \gamma_1(\beta_0)$, we compute: (we suppose that $\gamma_1(\beta_0) = [b] < [2]$)

$$0 = (\partial C)_{\alpha_0, \gamma_1(\beta_0) + \gamma_1(\beta_0), [1] + [2]} = - r[x_1, x_b] C_{\alpha_0 - [1] + \gamma_1(\beta_0), [1] + [2]} +$$

$$+ r[x_1, x_2] C_{\alpha_0, \gamma_1(\beta_0) + \gamma_1(\beta_0)} + \frac{\beta_0 !}{(\beta_0 - \gamma_1(\beta_0))!} [x_b, x_1] C_{\alpha_0, \beta_0}.$$

But now

$$\begin{aligned} 0 = (\partial C)_{\alpha_0, \gamma_1(\beta_0) + [2], [1] + [2]} &= \\ &= - (r-1)[x_1, x_2] C_{(\alpha_0 - [1] + \gamma_1(\beta_0)), [1] + [2]} \\ &- (r-1)[x_1, x_b] C_{\alpha_0 - [1] + [2], [1] + [2]} + \\ &+ r[x_1, x_2] C_{\alpha_0, \beta_0} + [x_2, x_1] C_{\alpha_0, \beta_0}. \end{aligned}$$

And:

$$\begin{aligned} 0 = (\partial C)_{\alpha_0, [2] + [2], [1] + [2]} &= \\ &= - (r-1)[x_1, x_2] C_{\alpha_0 - [1] + [2], [1] + [2]}. \end{aligned}$$

Thus:

$$C_{\alpha_0, \beta_0} = C_{\alpha_0 - [1] + \gamma_1(\beta_0), [1] + [2]}.$$

And:

$$\left(r + \frac{\beta_0 !}{(\beta_0 - \gamma_1(\beta_0))!} \right) [x_1, x_b] C_{\alpha_0, \beta_0} = r[x_1, x_2] C_{\alpha_0, \gamma_1(\beta_0) + \gamma_1(\beta_0)}$$

which proves our proposition in this case.

If $\beta_0 = [2] + [2]$, we have:

$$\begin{aligned} 0 = (\partial C)_{\alpha_0, [1] + [3], [2] + [2]} &= - r[x_1, x_3] C_{\alpha_0, \beta_0} + \\ &+ (r-1)[x_1, x_2] C_{\alpha_0 - [1] + [2], [1] + [3]} \\ &+ [x_1, x_2] C_{\alpha_0, [2] + [3]}. \end{aligned}$$

If $\beta_0 = [1] + [2]$, we consider:

$$0 = (\partial C)_{\alpha_0,[1]+[3],\beta_0} = - r[x_1, x_3]C_{\alpha_0,\beta_0} + \\ + (r-1)[x_1, x_2]C_{\alpha_0,[1]+[3]} \\ + [x_1, x_2]C_{\alpha_0,[1]+[3]} + [x_3, x_1]C_{\alpha_0,\beta_0}.$$

Our proposition is proved if dim $[\mathfrak{G}, \mathfrak{G}] > 1$.

Second case: dim $[\mathfrak{G}, \mathfrak{G}] = 1$, $[x_1, x_2] = x_2$:

In this case:

$$0 = (\partial C)_{\alpha_0,\beta_0,[1]} = \sum_j [x_1, x_j]D^{[j]}C_{\alpha_0,\beta_0} + C_{\alpha_0,\beta_0}C^2_{21} - C^2_{21}C_{\beta_0,\alpha_0}$$

$$= 2C_{\alpha_0,\beta_0} + \sum_j a_j x_2 D^{[j]}C_{\alpha_0,\beta_0}$$

if $[x_1, x_j] = a_j x_2$ where $a_j \in R$. Q.E.D. ■

COROLLARY 1. ($[\mathfrak{G}, \mathfrak{G}] \neq 0$). *If* $\mathfrak{G} \neq h_r \oplus \mathbb{R}^m$ *and* C *is a 2-cocycle with:*

$$F(C) = \{(\alpha, \beta) \quad s.t. \quad C_{\alpha,\beta} \neq 0\},$$

$$(\alpha_0, \beta_0) = \operatorname{Sup} F(C), \quad (|\alpha_0|, |\beta_0|) = (r, 2)$$

then there exists a one cochain B such that:

$$\operatorname{Sup} F(C - \partial B) < \operatorname{Sup} F(C).$$

Proof. With our hypothesis, we saw that $1 \in \operatorname{Supp} \alpha_0$, $2 \in \operatorname{Supp}\beta_0$, $\gamma_1(\beta_0) \leqslant \gamma_1(\alpha_0)$ and $C_{\alpha_0,\beta_0} = [x_1, x_2]f$ with $f \in N$. Let us consider:

$$B = fD^{\alpha_0 - [1] + \gamma_1(\beta_0)} \frac{(\alpha_0 - [1])!}{(\alpha_0 - [1] + \gamma_1(\beta_0))!}.$$

Then $\gamma_1(\alpha_0 - [1] + \gamma_1(\beta_0)) = \gamma_1(\beta_0)$, and

$$\operatorname{Sup} F(\partial B) = (\alpha_0, \beta_0)$$

$$(\partial B)_{\alpha_0,\beta_0} = f[x_1, x_2] = C_{\alpha_0,\beta_0}.$$ ■

THEOREM 3. *1. If* dim $[\mathfrak{G}, \mathfrak{G}] = 0$, *then:*

$$H^2_{\text{diff}}(N, \partial) = (\Lambda^2 S_+) \otimes N.$$

2. If dim $[\mathfrak{G}, \mathfrak{G}] \geqslant 1$, *and* $\mathfrak{G} \neq h_r \oplus \mathbb{R}^m$, *then:*

$$H^2_{\text{diff}}(N, \partial) \simeq I \oplus H^2(\mathfrak{G}, N).$$

3. If $\mathfrak{G} = h_r \oplus \mathbb{R}^m$*, and* $\dim [\mathfrak{G}, \mathfrak{G}] = 1$*, then:*

$$H^2_{\text{diff}}(N, \partial) = I \oplus H^2(\mathfrak{G}, N) \oplus (N(\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]) \otimes S_{++})$$

where $N(\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}])$ *is* $S(\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}])$ *(resp.* $C^\infty(\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}])$*) if* N *is* $S(\mathfrak{G})$ *(resp.* $C^\infty(\mathfrak{G})$*) and* S_{++} *is the ideal of* $S(\mathfrak{G})$ *spanned by* $\mathfrak{G} \cdot S_+$.

Proof. 1. is proved in proposition 1.

2. keeping our notations, by induction on Sup $F(C)$, we can write any 2-cocycle C as:

$$C = \lambda S^3 + C_{1,1} + \partial B$$

where $\lambda \in I$ and $C_{1,1}$ has order (1,1). We put

$$c(X, Y) = C_{1,1}(x, y) \qquad \forall X, Y \in \mathfrak{G}$$

$c \in Z^2(\mathfrak{G}, N)$ and the map:

$$\varphi : [C] \to (\lambda, [c]), \quad H^2_{\text{diff}}(N, \partial) \to I \oplus H^2(\mathfrak{G}, N)$$

is well defined:

$$[C] = [C'] \text{ implies } (\lambda - \lambda')S^3 + (C_{1,1} - C'_{1,1}) = \partial B,$$

thus since the left hand size has terms with maximal order (3,3) and the right one $(r, 2)$, the only possibility is:

$$\lambda = \lambda', \quad B = \sum_{|\alpha|=1} B_\alpha D^\alpha$$

and if: $\quad b(X) = B(x) \qquad \forall X \in \mathfrak{G}$

$$c - c' = \mathrm{d}b.$$

φ has an inverse mapping:

$$\psi : I \oplus H^2(\mathfrak{G}, N) \to H^2_{\text{diff}}(N, \partial)$$

$$(\lambda, [c]) \to [\lambda S^3 + C_{1,1}]$$

where $C_{1,1}$ is defined by:

$$C_{1,1} = \sum_{|\alpha|=|\beta|=1} C_{\alpha\beta} D^\alpha \cup D^\beta, \quad C_{1,1}(x, y) = c(X, Y).$$

ψ is obviously well defined.

3. Let us put $z = [x_1, x_2]$, $\mathscr{Z} = [\mathfrak{G}, \mathfrak{G}]$. We saw that in this case: if C is a 2-cocycle and $\operatorname{Sup} F(C) = (\alpha_0, \beta_0)$ $(|\alpha_0|, |\beta_0|) = (r, 2)$, there exists a 1-cochain B such that $\operatorname{Sup} F(C - \frac{1}{z}\,\partial B) < \operatorname{Sup} F(C)$($\frac{1}{z}\,\partial B$ is a 2-cocycle since $z \in I$ and $\frac{1}{z}\,\partial B$ is regular). Now we can write:

$$B_\alpha = zB_{1_\alpha} + B_{0_\alpha} \qquad \forall\alpha \quad \text{so that} \quad |\alpha| > 1$$

where $B_{0_\alpha} = B_\alpha|_{z=0}$ does not depend of the variable $z : B_{0_\alpha} \in N(\mathfrak{G}/\mathscr{Z})$.

Thus by induction on $\operatorname{Sup} F(C)$, we find:

$$C = \frac{\lambda}{z^3}\, S^3 + \frac{1}{z}\,\partial B_0 + C_{1,1} + \partial B$$

where $\lambda \in I$, $B_0 = \sum_{|\alpha|>1} B_{0_\alpha} D^\alpha$, $B_{0_\alpha} \in N(\mathfrak{G}/\mathscr{Z})$,

$$C_{1,1} = \sum_{|\alpha| = |\beta| = 1} C'_{\alpha\beta} D^\alpha \cup D^\beta.$$

We define a map $\varphi : H^2_{\text{diff}}(N, \partial) \to I \oplus H^2(\mathfrak{G}, N) \oplus (N(\mathfrak{G}/\mathscr{Z}) \quad S_{++})$ by:

$$\varphi([C]) = \left(\lambda, [c], \sum_{|\alpha|>1} B_{0_\alpha} \otimes X^\alpha\right)$$

where $c(X, Y) = C_{1,1}(x, y) \qquad \forall X, Y \in \mathfrak{G}$.

φ is well defined since $[C] = [C']$ implies:

$$\frac{\lambda - \lambda'}{z^3}\; S^3 + (C_{1,1} - C'_{1,1}) + \frac{1}{z}\; \partial(B_0 - B'_0) + \partial(B - B') = 0.$$

With our preceeding argument,

$$\lambda = \lambda', \quad \frac{1}{z}\,\partial(B_0 - B'_0) + \sum_{|\alpha|>1} \partial((B - B')_\alpha D^\alpha) = 0.$$

But if $\alpha_0 = \operatorname{Sup}\{\alpha/(B_0 - B'_0)_\alpha \neq 0\}$, we have:

$$\operatorname{Sup} F\left(\frac{1}{z}\; \partial(B_0 - B'_0)\right) = (\alpha_0 - \gamma_1(\alpha_0) + [1], \gamma_1(\alpha_0) + [2])$$

and:

$$\left(\frac{1}{z}\; \partial(B_0 - B'_0)\right)_{\alpha_0 - \gamma_1(\alpha_0) + [1], \gamma_1(\alpha_0) + [2]} =$$

$$= \frac{\alpha_0!}{(\alpha_0 - \bar{\gamma}_1(\alpha_0))!}\; (B_0 - B'_0)_{\alpha_0} \in N(\mathfrak{G}/\mathscr{Z}).$$

The equality is impossible if $B_0 \neq B'_0$. Thus $B_0 = B'_0$ and if $b(X) = (B - B')(x)$, $c - c' = - db$.

The inverse of φ is of course:

$$\psi : I \oplus H^2(\mathfrak{G}, N) \oplus (N(\mathfrak{G}/\mathscr{Z}) \otimes S_{++}) \to H^2_{\text{diff}}(N, \partial)$$

defined by:

$$\left(\lambda, [c], \sum_{|\alpha| > 1} B_{0_\alpha} \quad X^\alpha\right) \to \frac{\lambda}{z^3}\, S^3 + C_{1,1} + \frac{1}{z}\, \partial B_0$$

where

$$C_{1,1} = \sum_{|\alpha| = |\beta| = 1} C_{\alpha,\beta} D^\alpha \cup D^\beta,$$

$$C_{1,1}(x, y) = c(X, Y), \quad B_0 = \sum_{|\alpha| > 1} B_{0_\alpha} D^\alpha .$$

■

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